CYCLIC COMMUTATIVITY NEAR RING II

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Abstract

A right near-ring N is called weak commutative $if\ xyz = xzy\ for\ all\ x, y, z \in N$. A right near-ring N is called pseudo commutative $if\ xyz = zyx\ for\ all\ x, y, z \in N$. A right near-ring N is called quasi-week commutative $if\ xyz = yxz\ for\ all\ x, y, z \in N$. It is quite natural to investigate the properties of a near- ring (or a ring) N with the property $xyz = yzx\ \ for\ every\ x, y, z \in N$.We call such a near-ring as cyclic commutative. We define and obtain many interesting properties of cyclic commutativity.

Keywords: *Algebra; Commutativity; Cyclic Commutativity: Near rings.*

1. INTRODUCTION

Herstien [9] proved that if R is a ring such that for each $x, y \in R$, there exists a natural number $n = n\,(x,y) > 1$ depending on x, y with $(xy - yx)^n = xy - yx$, then R is a commutative ring. H.E.Bell [1] proved that if R is a distributively generated near – ring with unity 1 and if $for\ each\ \ x, y \in R$, there exists an integer $n = n\,(x,y) > 1$ with $(xy - yx)^n = xy - yx$ then R is a commutative ring. Desmond Machale [12] proved that if R is a ring such that $for\ each\ \ x, y \in R$, there exists an even, natural number $n = n(x, y)$ depending $x\ and\ y$ with $(xy + yx)^n = xy + yx$, t hen R is anti-commutative.

K. Chandrasekar Rao and G.Gopalakrishnamoorthy [3] proved that if R is a distributively generated near – ring with unity 1 such that $(xy + yx)^n = (xy + yx)\,for\ all\ x, y \in R$ where $n = n(x, y) > 1$ is an integer depending on $x\ and\ y$, then R is a commutative ring .In this paper we define a ring R to be cyclic commutative if $xyz = yzx$ for all $x, y, z \in R$ and try to generalize the results of Herstein, H.E.Bell, Desmond Machale and G.Gopalakrishnamoorthy. Luckily we have obtained many beautiful and interesting results.

2. DEFINITIONS AND BASIC PROPERTIES

Definition 2.1

Let R be a ring, near-ring R is said to be cyclic commutative $if\ xyz = yzx\ for\ all\ x, y, z\ \in R$

R is said to be cyclic anti-commutative $if\ xyz = -yzx\ \ for\ all\ x, y, z \in R$

Example 2.2

Every commutative ring is cyclic commutative.

Definition 2.3

A ring R is called a prime ring $if\ xRy = 0\ implies\ either\ x = 0\ or\ y = 0$

$$(ie)\, xry = 0\ \forall\, r \in R\ \ implies\ either\ x = 0\ or\ y = 0$$

Definition 2.4

A ring R is called a semi-prime ring $if\ xRx = 0\ implies\ \ x = 0$

$$(ie)\, xrx = 0\ \forall\, r \in R\ \ implies\ \ x = 0$$

Note 2.5

Every prime ring is a semi-prime ring

Definition 2.6

A ring R is called a primitive $if\ R\ has\ a\ maximal\ right$ ideal I which contains no non-zero ideal of R

Definition 2.7

A ring R is said to be semi-simple (in the sense of Jacobson) if it is isomorphic to a sub direct sum of primitive rings R_α , each of which is a homomorphic image of R.

Definition 2.8

Let R be a ring with unity 1.Then the set $J = \{r \in R \backslash 1 - r\ is\ inivertible\ in\ R\}$ is called the Jacobson radical of R

Definition 2.9

Let R be a ring. The intersection of all maximal ideals of R is called the radical of R.

Lamma 2.10

Let R be a ring in which $xyz = 0$ implies $yzx = 0$ If $e \in R$ is idempotent then e lies in the center of R

Proof

let $x \in R$ be an arbitrary element:

Then $exe - e^2xe = 0$

$e(x - ex)e = 0$

So, $(x - ex)ee = 0$

$(x - ex)e = 0$

$xe - exe = 0$

$\therefore xe = exe \quad \forall x \in R.$ (1)

By (i) $xe - exe = 0$

$(x - exe)e = 0$

$(x - exe)e.e = 0$

$ee(x - exe) = 0$

$e(x - exe) = 0$

$ex - exe = 0 \therefore ex = exe \quad \forall x \in R$ (2)

(1) and (2) gives

$ex = xe \quad \forall x \in R$

Thus e lies in the center of R

Lemma 2.11

Every Boolean ring R is cyclic commutative ring

Proof

Let R be a Boolean ring, For all $x, y, z \in R, we\ have$

$$(x + yz)^2 = (x + yz)$$

$$x^2 + xyz + yzx + (yz)^2 = x + yz$$

$$x + xyz + yzx + yz = x + yz$$

$So,\ xyz + yzx = 0\ for\ all\ x, y, z \in R$ (3)

So, R is cyclic anti commutative

Taking $y = z = x$ in 1 we get $x^3 + x^3 = 0$ (ie) $x + x = 0$

$$x = -x \quad for\ all\ x \in R \tag{4}$$

From (3) and (4) we get

$$xyz = yzx \qquad \forall\, x, y, z \in R$$

(ie) R is cyclic commutative ring

Lemma 2.12

Let R be a ring in which $x^2 \in Z(R)$ $for\ all\ x \in R$.where $Z(R)$ is the centre of R.

$Then$, $(xyz)^2 = (yzx)^2$ $for\ all\ y, z \in R$.

Proof

For any $x, y, z \in R$, $we\ have$

$$(xyz)^2 = xyz.xyz = xuxu \quad where\ u = yz$$

$$= x(uxu)$$

$$= x\{(xu)^2 + u^2 - (xu - u)^2 - xu^2\}$$

$$= \{x(xu)^2 + xu^2 - x(xu - u)^2 - x^2u^2\}$$

$$= \{(xu)^2x + u^2x - (xu - u)^2x - u^2x^2\}$$

$$= \{(xu)^2 + u^2 - (xu - u)^2 - u^2\}x$$

$$= (uxu)x$$

$$= (ux)^2$$

$$= (yzx)^2$$

$$(xyz)^2 = (yzx)^2 \qquad \forall\ x, y, z \in R$$

Lemma 2.13

Let R be a ring such that

$$x^2yz = yzx^2 \ \ for\ all\ x, y, z \in R.Then\ \ (xyz)^2 = (yzx)^2\ for\ all\ x, y, z \in R$$

Proof

Let $x, y, z \in R$.

$$(xyz)^2 = (xyz)(xyz) = (xyzx)yz$$

$$= (xux)yz \qquad where\ u = yz$$

$$= \{(ux)^2 + x^2 - (ux - x)^2 - ux^2\}yz$$

$$= (ux)^2yz + x^2yz - (ux - x)^2yz - ux^2yz$$

$$= yz(ux)^2 + yzx^2 - yz(ux - x)^2 - uyzx^2$$

$$= yz(ux)^2 + yzx^2 - yz(ux - x)^2 - yzux^2$$

$$= yz\{(ux)^2 + x^2 - (ux - x)^2 - ux^2\} \quad (u = yz)$$

$$= yz(xux)$$

$$= yz(xyzx)$$

$$= (yzx)(yzx) = (yzx)^2$$

Hence proved

3. MAIN RESULTS

Theorem 3.1

Let D be a division ring in which for every $x, y, z \in D$ there exists an integer $n = n(x, y, z) > 1$ Such that $(xyz - yzx)^n = xyz - yzx$. Then is D is a cyclic commutative.

Proof

$If\ xyz - yzx = 0\ ,\ there\ is\ nothing\ to\ prove\ .$

So, assume that $for\ some\ a, b, c \in D, abe - bca \neq 0$.

$Let\ \ Z\ \ be\ the\ center\ of\ D. If\ \lambda \in Z, then\ \lambda(abc - bca) = (\lambda a)bc - bc(\lambda a).$

Then by the hypothesis, there exists positive integers $n = n(a, b, c) > 1\ and\ m = m(\lambda a, b, c) > 1$ $such\ that$

$$(abc - bca)^n = abc - bca \tag{5}$$

$$and\ ((\lambda(abc - bca))^m = \lambda(abc - bca) \qquad \therefore \lambda \in Z \tag{6}$$

Let $k = k(\lambda) = (n - 1)(m - 1) + 1.\ Then\ \ k > 1\ and$

$$(abc - bca)^k = abc - bca \tag{7}$$

$$and\ \ (\lambda(abc - bca))^{k.} = \lambda(abc - bca) \tag{8}$$

Since $\lambda \in Z,\ (4)\ becomes$

$$\lambda^k(abc - bca))^{k.} = \lambda(abc - bca) \tag{9}$$

Using (7) we get

$$\lambda^k(abc - bca) = \lambda(abc - bca)$$

Since D is a division ring and $-bca \neq 0,\ abc - bca$ *is invertible in D* .

So, from (9) we get $\lambda^k = \lambda$.

This shows that for every $\lambda \in Z$, there exists an integer $k = k(\lambda) > 1$

Such that $\lambda^k = \lambda$ (10)

But then Z must be a field of characteristic $p \neq 0$. Moreover Z is algebraic over its prime field P, which has p elements. *Let* $u = abc - bca \neq 0$.

Since $u^k = u, u$ is algebraic over P and so algebraic over Z.

$$a \notin Z \ for\ if\ a \in Z,$$

Now, $au = a(abc - bca)$

$= aa(bc) - a(bc)a$

$If\ u \in Z\ ,then\ au \notin Z, for\ if\ au \in Z, then\ a \in Z$

And so $abc - bca = 0, contradictive\ our\ assumption.$

Without loss of generality we may assume $u \notin Z\ for\ if\ u \in Z$, we would carry our argument for au rather than u.

Consequently u satisfies a minimal polynomial

$x^t + \lambda_1 x^{t-1} + \lambda_2 x^{t-2} \dots \dots \dots \dots \dots + \lambda_t = 0$ over Z *of degree* $t > 1$,

Let $F = P(\lambda_1, \lambda_2 \dots \dots \dots \dots \dots \lambda_t)$.be the smallest field, containing F and $\lambda_1, \lambda_2 \dots \dots \dots \dots \dots \lambda_t$. Because λ_i are algebraic over P and commute with each other, F is a finite field and has , say, q elements

Clearly $\boldsymbol{if\ w\omega \in F,}$ then $w^q = w\ \boldsymbol{\omega}$.

Consider the field $F(u)$. The polynomial $x^q - x$ clearly has q roots in F and since it can have at most q roots in (u) , $u \notin F \subset Z$

We can conclude that $u^q \neq u$ (11)

Now $u^t + \lambda_1 u^{t-1} + \lambda_2 u^{t-2} + \lambda_3 u^{t-2} + \cdots \dots \dots \dots \dots + \lambda_t = 0$ (12)

So, $0 = (u^t + \lambda_1 u^{t-1} + \lambda_2 u^{t-2} + \lambda_3 u^{t-2} + \cdots \dots \dots \dots \dots + \lambda_t)^q$

$= u^{qt} + {\lambda_1}^q u^{q(t-1)} + {\lambda_2}^q u^{q(t-2)} \dots \dots \dots \dots \dots + {\lambda_t}^q$

$= (u^q)^t + \lambda_1 (u^q)^{t-1} + \lambda_2 (u^q)^{t-2} \dots \dots \dots \dots \dots \dots + \lambda_1$ $(using\ (10))$

Thus $u\ and\ u^q$ are both roots of the same polynomial over Z. This implies that there is an element $r \in D$ *such that* $u^q = rur^{-1}$

That is $ru = u^q r$ (13)

$Consequently\ ru \neq ur$.(For if $ru = ur, then\ ur = u^q r$

and so $u = u^q r r^{-1} = u^q\ contradicting$ (11)

Let $y = uru - ru^2 = uru - ruu$

Then $yu = (uru - ru^2)u$

$= (uu^q r - u^q ru)u$

$= u^q(uru - ru^2)$

$$yu = u^q y \quad (14)$$

By the hypothesis, there exists an integer $t = t(u, r, u) > such\ that$

$$y^l = y \quad (15)$$

$$Let \quad T = \left\{ \sum_{i=0}^{t-1} \sum_{j=0}^{n-1} p_{ij} y^i u^j / p_{ij} \in P \right\}$$

T is clearly finite and is an additive subgroup of D.

$$Since\ yu = u^q y, T\ is\ also\ closed\ under\ multiplication\ .$$

Hence T is a finite division ring. By Wedderburn's Theorem, it follows that T is a commutative field. But both $u\ and\ y\ are\ in\ T$.

So $yu = uy$.

$$Since\ yu = u^q y\ \ we\ get\ uy = u^q y\ \ and\ so\ we\ get\ \ u = u^q$$

which is contradicts $u^q \neq u$,

This contradiction proves that

$$xyz - yzx = 0 \qquad for\ all\ x, y, z \in D$$

Lemma 3.2

If A is a prime ring in which for every $x, y, z \in A$, there exists an integer $n = n(x, y, z) > 1$ such that, $(xyz - yzx)^n = xyz - yzx$. Than A has no non-zero nilpotent elements

Proof

Suppose A has a non-zero nilpotent elements, it has an element $x \neq 0\ such\ that\ x^2 = 0$

If $r \in A$ is any element, we have $xrx = xrx - rxx$

By hypothesis, there is an integer $n = n(x, r, x) > 1\ \ such\ that$

$$(xrx - rxx)^n = xrx - rxx$$

(ie) $(xrx)^n = xrx$ (16)

Now $(xrx)^2 = xrx.xrx = xrx^2rx = 0$

And consequently $(xrx)^n = 0$. Then from (16) we get

$$xrx = 0 \qquad \forall\ r \in A$$

Since A is semi prime, $x = 0$

Hence A has no non-zero nilpotent element

Lemma 3.3

If A is any prime ring in which for every $x, y, z \in A$, there exists an integer $n = n(x, y, z) > 1$ *such that* $(xyz - yzx)^n = (xyz - yzx)$. Then any idempotent in A is in the center of A

Proof

Let $e \in A$ be any idempotent element

Then $e^2 = e$ Now $(xe - exe)^2 = (xe - exe)(xe - exe)$

$$= xexe - xe^2xe - exexe + exe^2xe$$

$$= xexe - xexe - exexe + exexe$$

$$= 0$$

Since A has no non – zero nilpotent element, we get $xe - exe = 0$

(ie) $xe = exe$

III^{ly} $(ex - exe)^2 = 0$ and so $ex = exe$

Thus $ex = xe$ *for all* $x \in A$

Hence e belongs to the center of A

Theorem 3.4

Let A be a primitive ring in which for every $x, y, z \in A$, there exists an integer $n = n(x, y, z) > 1$ *such that* $(xyz - yzx)^n = xyz - yzx$. Then A is cyclic commutative.

Proof

Since A is a primitive ring, it has a maximal right ideal I which contains no non – zero two – sided ideal of A.

Thus $I \wedge Z = (0)$ *where z is a center of A, for if* $I \wedge Z \neq (0)$ then there exists a non –zero

$x \in I \wedge Z$, *. Then* $xA = Ax \subset I$ is a two – sided ideal of A which contain in I.

So $xA = Ax = (0)$

Since A is primitive, we get $x = 0$.

Let $x, y, z \in I$ $By\ the\ hypothesis\ there\ exists\ a\ positive\ integer$

$$n = n(x, y, z) > 1\ such\ that\ (xyz - yzx)^n = (xyz - yzx)$$

Let $e = (xyz - yzx)^{n-1} \in I$.Then $e^2 = e$ .By lemma 3.3 $e \in Z$.

That is $e \in I \wedge Z$

Since $I \wedge Z = (0)$ we get $e^2 = 0$

Thus $0 = e(xyz - yzx)$

$= (xyz - yzx)^{n-1}(xyz - yzx)$

$= (xyz - yzx)^n$

$= xyz - yzx$

(ie) $xyz - yzx = 0\ for\ all\ x, y, z \in I,$

Theorem 3.5

If A be a semi-simple ring in which for every $x, y, z \in A$, there exists an integer $n = n(x, y, z) > 1$ such that $(xyz - yzx)^n = xyz - yzx$ .Then A is cyclic commutative.

Proof

Let A is semi-simple ring,

In the sense of Jacobson, A is isomorphic to a sub-direct sum of primitive rings, A_α.

Each of these primitive rings A_α is a homomorphic image of A and so inherits the property that for every $x, y, z \in A_\alpha$, there exists an integer $n = n(x, y, z) > 1$ such that

$$(xyz - yzx)^n = xyz - yzx.$$

By theorem 4, these primitive rings A_α are cyclic commutative rings and so A is cyclic commutative.

Theorem 3.6

Let A be a ring in which for every $x, y, z \in A$, there exists an integer $n = n(x, y, z) > 1$ Such that, $(xyz - yzx)^n = xyz - yzx$. Then A is cyclic commutative

Proof

Let N be the radical of A . Then A/N is semi –simple and so by theorem 5

A/N is cyclic commutative.

Thus $xyz - yzx \in N$ $for\ all\ x, y, z \in A$. More over there exists an integer

$n = n(x, y, z) > 1$ Such that, $(xyz - yzx)^n = xyz - yzx$

Let $e = (xyz - yzx)^{n-1}$

Then $e^2 = e$ says e is an idempotent and $e \in N$. But the only idempotent element in the radical is 0. So $e = 0$

(ie) $0 = e(xyz - yzx)$

$= (xyz - yzx)^{n-1}(xyz - yzx)$

$= (xyz - yzx)^n$

$= xyz - yzx$

(ie) $xyz - yzx = 0 \quad \forall\, x, y, z \in A$

Then A is cyclic commutative.

Theorem 3.7

Let R be a ring in which $(xyz + yzx)^2 = xyz + yzx$ $for\ all\ x, y, z \in R$. Then, R is anti-cyclic commutative (ie) $xyz + yzx = 0$ $for\ all\ x, y, z \in R$.

Proof

Suppose that $xyz = 0$

Then $yzx = xyz + yzx$

$= (xyz + yzx)^2$

$= (yzx)^2 = yzxyzx$

$yzx = yz(xyz)x = 0$

Hence by lemma 2.10, every idempotent element is central.

So $(xyz + yzx) \in Z(R)$ (17)

Hence $x(xyz + yzx) = (xyz + yzx)x$

(ie) $x^2yz + xyzx = xyzx + yzx^2$

(ie) $x^2yz = yzx^2$ $for\ all\ x, y, z \in R$ (18)

Then by lemma 2.13, $(xyz)^2 = (yzx)^2$ (19)

Also $yz(xyz + yzx) = (xyz + yzx)yz$

$yzxyz + (yz)^2x = x(yz)^2 + yzxyz$

$(yz)^2x = x(yz)^2$ $for\ all$ $x, y, z \in R$ (20)

$$Now\ (-xyz - yzx) = (-xyz - yzx)^2$$

$$= (xyz + yzx)^2$$

$$= (xyz + yzx)$$

$$(ie) \quad 2\ (xyz + yzx) = 0 \tag{21}$$

$$Now\ (xyz + yzx)^2 = (xyz)^2 + (yzx)^2 + (xyz)(yzx) + (yzx)(xyz)$$

$$= (xyz)^2 + (xyz)^2 + x(yz)^2x + (yz)x^2(yz) \quad (using\ (19))$$

$$= 2(xyz)^2 + x.x(yz)^2 + x^2(yz)(yz) ---------- using\ (18)\ and\ (20)$$

$$= 2(xyz)^2 + 2x^2(yz)^2$$

$$(xyz + yzx)^2 = 2\{(xyz)^2 + x^2(yz)^2\} \tag{22}$$

Now $(xyz + yzx) = (xyz + yzx)^2$

$$multiplying\ by\ xyz + yzx$$

$$(xyz + yzx)^2 = (xyz + yzx)^3$$

$$(xyz + yzx) = (xyz + yzx)^2.(xyz + yzx)$$

$$= 2\{(xyz)^2 + x^2(yz)^2\}(xyz + yzx)$$

$$= \{(xyz)^2 + x^2(yz)^2\}2(xyz + yzx)$$

$$(xyz + yzx) = 0 \quad for\ all\ x,y,z \in R \quad using\ (21)$$

Hence R is anti –cyclic commutative

Lemma 3.8

Let R be a ring. Suppose for any $a, b \in R$, there exists positive integers

$$n(a) > 0\ and\ n(b) > 0\ such\ that\ a^{n(a)} = a\ and\ a^{n(b)} = b, then\ a^t = a\ \ and\ b^t = b$$

$$where \quad t = (n(a) - 1)(n(b) - 1) + 1$$

More over if $n(a)$ and $n(b)$ are even then t is even

Proof

$$a^t = a^{(n(a)-1)(n(b)-1)+1}$$

$$= a^{(n(a)-1)(n(b)-1)+1}.a$$

$$= a^{n(a)n(b)-n(a)-n(b)+2}$$

$$= a^{n(a)n(b)}a^{-n(a)}a^{-n(b)}a^2$$

$$=(a^{n(a)})^{n(b)} a^{-n(a)} a^{-n(b)} a^2$$

$$= a^{n(b)} a^{-1} a^{-n(b)} a^2$$

(ie) $a^t = a$

Similarly $b^t = b$. More over t is even $if\ n(a)\ and\ n(b)\ are\ both\ even$

Lemma 3.9

Let R be a ring in which $xyz = 0$ $implies\ yzx = 0. Then\ a^k r = 0\ implies\ (ar)^k = 0$ $for\ all\ a, r \in R\ and\ for\ all\ positive\ integer\ k \geq 1$Also $ra^k = 0\ implies\ (ra)^k = 0$ $for\ all\ a, r \in R$

Proof

Let $a, r \in R$ and $k \geq 1$ be any integer

Now $a^k r = 0 \Rightarrow a^{k-1} ar = 0$

$\Rightarrow ara^{k-1} = 0$ (by the hypothesis of theorem)

$\Rightarrow ar\ a^{k-1} r = 0$

$\Rightarrow ar\ a^{k-2} ar = 0$

$\Rightarrow ar\ (a^{k-2} ar) = 0$

$\Rightarrow ar. ar\ a^{k-2} = 0$

$\Rightarrow (ar)^2\ a^{k-2} r = 0$

$\Rightarrow (ar)^2\ a^{k-3} ar = 0$

$\Rightarrow ((ar)^2\ a^{k-3}) ar = 0$

$\Rightarrow a^{k-3}(ar)(ar)^2 = 0$

$\Rightarrow a^{k-3}(ar)^3 = 0$

$Proceeding\ we\ get\ (ar)^k = 0 \quad \forall\ a, r \in R\ and\ for\ all\ integer\ k \geq 1$

Similarly $ra^k = 0\ implies\ (ra)^k = 0$

Theorem 3.10

Let R be a ring with unity in which for every $x, y, z \in R$, there exists an even integer $n = n(x, y, z) \geq 2$ $depending\ on\ x, y, z\ such\ that$ $(xyz + yzx)^n = xyz + yzx$

Then R is anti- cyclic commutative

Proof

By lemma 3.8, there exists an even integer $n = n(x, y, z) = n(-x, y, z) \geq 2$

Such that $(xyz + yzx)^n = (xyz + yzx)$ (23)

Then $(-xyz - yzx)^n = (-xyz - yzx)$ (24)

Since n is even , $(xyz + yzx)^n = (-xyz - yzx)^n$

Then $xyz + yzx = (xyz + yzx)^n = (-xyz - yzx)^n$

$= -xyz - yzx$

$\therefore$ $2(xyz + yzx) = 0$ (25)

Taking $y = z = 1,\ in\ (25)\ we\ get$

$$4x = 0$$

Taking $y = z = 1,\ in\ (1) we\ get$

$$2x = (2x)^n = 2^n x^n = 0\ \ if\ n \geq 2$$

Thus R is of characteristic 2 .We know a ring R of characteristic 2 is anti cyclic commutative if R is cyclic commutative. In such a ring, the hypothesis of the theorem reduces to that of theorem 3.6 and the result follows from theorem 3.6.

Theorem 3.11

Let R be a near-ring (left) with unity 1.suppose that for any $x, y, z \in R$ there exists a positive integer $n = n(x, y, z) > 1\ depending\ on\ x, y, z\ with\ (xyz + yzx)^n = xyz + yzx$.

$Then\ \ ayz + yza = 0\ for\ all\ y, z \in R\ and\ for\ all\ a \in N$, where N is the set of all nilpotent elements of R.

Proof

If $xyz = 0$, then for any integer $m > 1$

$(xyz + yzx)^m = (yzx)^m = (yzx)^2 (yzx)^{m-2}$

$= (yzx)(yzx)(yzx)^{m-2}$

$= yz(xyz)\, x(yzx)^{m-2}$

$=0$

(ie) $xyz = 0\ \ implies\ (xyz + yzx)^m = 0\ for\ all\ m > 1$ (26)

Now by the hypothesis of the theorem, there exists a even integer $n = n(x, y, z) > 1\ \ such\ that\ \ (xyz + yzx)^n = xyz + yzx$ (27)

From (1) and (2) we get

$$xyz + yzx = 0$$

(ie) $yzx = 0$

(ie) $xyz = 0$ $implies$ $yzx = 0$ $for\ all\ x, y, z \in N$ (28)

Let $a \in N$, where N is the set of all nilpotent elements of R. We shall prove by induction on the degree of nilpotence of a that $ayz + yza = 0$ $for\ all\ y, z \in R$

Suppose that $a^2 = 0$

Then $a^2 r = 0$ $for\ all$ $r \in R$

(ie) $aar = 0. This\ implies\ ara = 0\ for\ all\ r \in R$ $(using\ 28)$ (29)

Now, $ayz(ayz + yza) = (ayza)yz + a(yz)^2 a = 0$ (using 29) (30)

Also $yza(ayz + yza) = yza^2 yz + yza(yz)a = 0$ (using 29 and $a^2 = 0$) (31)

(30) + (31) gives $(ayz + yza)^2 = 0$ (32)

Hence, $(ayz + yza)^m = 0$ $for\ all\ m > 1$

Now by the hypothesis of the theorem, there exists an integer $n = n(a, y, z) > 1$ such that

$$(ayz + yza)^n = (ayz + yza) \quad (33)$$

From (32) and (33) we get, $(ayz + yza) = 0$ $for\ all\ y, z \in R\ and\ for\ all\ a \in N\ with\ a^2 = 0$ (34)

Assume that,

$ayz + yza = 0$ $for\ all\ y, z \in R\ and\ for\ all$ $a \in N$ whose degree of $nilpotence\ is\ less\ then\ n.$ (35)

Let $a \in N$ such that the degree of nilpotence is n.

Then $a^n = 0$

Now $a^n = 0$ $implies$ $(a^{n-1})^2 = a^{2n-2} = (a^n)^2 a^{-2} = 0$

Hence a^{n-1} is nilpotent whose degree of nilpotence is 2.

Hence a^{n-1} satisfy the equation (34)

(ie) $a^{n-1} yz + yza^{n-1} = 0$ $for\ all\ y, z \in R$ (36)

If z is replaced by za we get

$$a^{n-1} yza + yza^n = 0 \quad for\ all\ y, z \in R \quad (37)$$

Since $a^n = 0, we\ have\ a^n yz = 0 = yza^n$

Hence $a^{n-1} yza + a^n yz = 0$

$(ie) a^{n-1}(yza + ayz) = 0$ (38)

By Lemma 3.9 we get $\left(a(yza + ayz)\right)^{n-1} = 0$

So, $a(yza + ayz) = 0$

$\Rightarrow a(yza + ayz)yz = 0$

$\Rightarrow (yza + ayz)yza = 0$ (39)

Also, if $if\ y\ is\ replaced\ by\ ay\ in\ (36) we\ get$

$a^n yz + ayza^{n-1} = 0$ (40)

Since $a^n = 0, we\ have\ a^n yz = yza^n = 0$

Hence $yza^n + ayza^{n-1} = 0$

$(yza + ayz)a^{n-1} = 0$

By Lemma 3.9 $((yza + ayz)a)^{n-1} = 0$

So $(yza + ayz)a) = 0$

$\Rightarrow yz(yza + ayz)a = 0$

$\Rightarrow (yza + ayz)ayz = 0$ (41)

(39) + (41) gives

$(ayz + yza)^2 = 0$

$Hence\ (ayz + yza)^m = 0\ for\ all\ m > 1$ (42)

Now by the hypothesis of the theorem, there exists an integer

$n = n(a, y, z) > 1$ such that $(ayz + yza)^n = (ayz + yza)$ (43)

From (42) and (43) we get

$(ayz + yza) = 0\ y, z \in R\ and\ for\ all\ a \in N.$

Hence theorem

Lemma 3.12

Let R be a near – ring with unity 1. Suppose that for any $x, y, z \in R$ there exists a positive integer $n = n(x, y, z)\ depending\ on\ x, y, z\ with\ (xyz + yzx)^n = xyz + yzx$.

Then the nilpotent elements of R are distributive

Proof

Let $a \in N$

Then $(-a)(-a) = -((-a)a)$ (44)

(In a left near ring $x.(-y) = -xy$)

By Theorem 3.11, $ayz + yza = 0 \quad \forall\, y, z \in R$

Taking $z = 1, y = -a \quad we\ get$

$a(-a) + (-a)a = 0$

which gives $a(-a) = -((-a)a)$ (45)

Form (44) and (45) we get

$(-a)^2 = a(-a)$ (46)

Also $0 = a.0 = a(-a + a)$

$= a(-a) + a^2$

Which gives

$$a(-a) = -a^2 \quad (47)$$

(46) and (47) gives

$(-a)^2 = -a^2$

This implies $(-a)^n = -a^n$ $for\ all\ integer\ n \geq 1$

Hence $a \in N\ \ implies\ -a \in N$

Hence by Theorem 3.11

$(-a)yz + yz(-a) = 0 \quad \forall\, y, z \in R$

Taking $z = 1\ \ we\ get$

$(-a)y + y(-a) = 0 \quad \forall\ y \in R$

This implies

$$(-a)y = -y(-a) = ya \quad \forall\ y \in R \quad (48)$$

So, for all $x, y \in R$

$$(x + y)a = (-a)(x + y) \qquad (using 48)$$

$$= (-a)x + (-a)y \qquad (left\ near - ring)$$

$$= xa + ya \qquad ((using\ (5))$$

Thus $(x + y)a = xa + ya \qquad \forall\ x, y\ \in R\ and\ \ for\ all\ \ a \in N$

Lemma 3.13

Let R be a distributively generated near-ring with unity. Suppose that for every $x, y, z \in R$ there exists a positive integer

$n = n(x, y, z)$ *depending on* x, y, z *with* $(xyz + yzx)^n = xyz + yzx$. *Then* $(R, +)$ *is abelian* .

Proof

Let $a \in N$ be arbitrary By Theorem 3.11

$$ayz + yza = 0 \qquad \forall\, y, z \in R$$

Taking $y = z = 1$, *we get*

$a.1 + 1.a = 0 \qquad \forall\ a \in N$

(ie) $a + a = 0$ *for all* $a \in N$

(*ie*) $2a = 0$ $\forall a \in N$ .That is N is of characteristic 2

Then , (N,+) is abelian

Let $a, b \in N$. Then $a + b = b + a \qquad \forall\, a, b \in N$

Also by the hypothesis

$$(xyz + yzx)^n = xyz + yzx \qquad \forall\ x, y, z \in R$$

$$where\ \ n = n(x, y, z) > 1.$$

Take $x = y = z = 1$, *we get*

$$(1 + 1)^n = 1 + 1 \qquad for\ some\ \ n > 1$$

So, $(R, +)$ is torssion. If R is sub –directly irreducible, then $(R, +)$ is a $p -$group where p is a prime number.

Putting 2, *we get* $(1 + 1) \in N$ *This implies* $(1 + 1)$ *is distributive*

So $(R, +)$ is abelian. (Hence R is a ring)

Theorem 3.14

Let R be a distributively generated near – ring with unity 1.If for every $x, y, z \in R$,there exists an integer $n = n(x, y, z) > 1$ *such that* $(xyz + yzx)^n = xyz - yzx$ .Then

R is a cyclic commutative ring

Proof

By lemma 3.13, $(R, +)$ is an abelian group. Hence R ia a ring cyclic commutativity follows from Theorem 3.6

REFERENCE

[1] Bell.H.E, Certain near-ring are rings, J.London Math.Soc.4(2),264-270,(1971)

[2] Bell.H.E, Quasi-centres, quasi-commutators, and ring commutativity, Acta Math. Aug.41,127-136(1983)

[3] Chandra Sekhara Rao .K. Gopalakrishnamoorthy.G , Certain near-rings are commutative rings, pure and applied mathematics sciences,

[4] Frohlich.A. Distributively generated near–rings I –Ideal Theory Proc.London.Math.soc.8(3),76-94(1958).

[5] Gopalakrishnamoorthy.G, Kamaraj.M and Geetha.S, On Quasi weak commutative near – rings, Int.Jou.of Math.research, Vol 5(5) ,431-440,(2013)

[6] Gopalakrishnamoorthy.G, Geetha.S, and Anitha.S, On Quasi commutative near – rings-II, Malaya Jour of Mathematik,Vol 3(3), 327-334,(2015)

[7] Gopalakrishnamoorthy.G, Geetha.S, and Anitha.S,, On Quasi – Weak commutative Boolean – like near – rings,Malaya Jour of Mathematik, Vol 3(3), 318-326,(2015).

[8] Herstein I.N. Noon- commutative rings, Cares Mathematical Monographics, No 15, Mathematical association of America, Washington,D.C (1968)

[9] Herstein I.N, Condition for the commutativity of rings, cauad. J.Maths, 9(1957), 583-585

[10] Jacobson.N, Structure theory for algebraic algebras of bounded degree, Ann.math, 46(1945), 693-707

[11] Ligh.s. On distributively generated near-rings, Proc.Edinburgh.Math.Soc.16, 239-343(1969)

[12] Desmond Machale, An anti- commutativity consequence of a ring commutativity theorem of herstien. American Math.Moorthy 94,162-165(1987)

[13] Pilz Gunter, Near rings, North.Holhand, Amsterdaus(1977)

[14] Uma.S,Balakrishnan and Tamilchelvan.T, Pseudo commutative near-rings, Scientia Magna 6, No2,75-85(2010)