Diffusion Operandi Gradient Solutions

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Abstract: Moving away from the usual utilization of Topological Derivative like boundary related constraint this Domain based method using perturbation techniques provides a enhanced articulation on segmentation of medical images for recognition of abnormality in diseased organs. It is achieved with detailed clarity on edges in the medical images apart from the sectored part of the organs. The method discussed in this paper takes a step forward by introducing diffusion for the pixels concerned. This takes care of perturbation issue. Further by solving for the characteristic polynomial for the obtained matrix that represents the medical image, we will be finding the eigenvalues and eigenvectors. But after this diffusion is introduced in a two separate entity as an initial one and an final one. The after the introduction of tensors to take diffusion parameters and values, further calculation gets carried and the adjoint is found. Now the Topological gradient gets calculated as a function of difference in pixels. After this application of Sobolev space besides the presence of other domain related space along with the weak derivative helps us to arrive the desired results.

Keywords: Diffusion tensor, Topological derivative, Sobolev space, Weak derivative

I. INTRODUCTION

Besides the usual problem of restoration in the field of image processing, segmentation issues gets addressed in the field of medical image processing with Topological Derivative playing a role among the many mathematical techniques discussed here that finally helps one to bring out meticulous nature of the affected region of interest. Usual principle of making use of Topological Asymptotic expansion

$$I_\varepsilon(\Omega_\varepsilon) = I(\Omega) + f(\varepsilon) D_T(\hat{x}) + o(f(\varepsilon)) \qquad (1(a))$$

Where $I_\varepsilon(\Omega_\varepsilon)$ is the perturbed region in the medical image,

$I(\Omega)$ represents the domain of the concerned medical image,

$f(\varepsilon)$ is the monotonic function and is positive with $\varepsilon \to 0$,

$D_T(\hat{x})$ is the Topological derivative concentrated at the point $\hat{x}$,

$o(f(\varepsilon))$ forms the remainder of terms in the asymptotic expansion

Solving $(1(a))$, we will be able to obtain Topological derivative $D_T(\hat{x})$ which will pave way for calculating the cost function associated in our problem

$$D_T\left(\hat{x}\right)=\lim_{\varepsilon\to 0}\frac{I_\varepsilon\left(\Omega_\varepsilon\right)-I(\Omega)}{f\left(\varepsilon\right)} \qquad \left(1\left(b\right)\right)$$

Equation $\left(1\left(b\right)\right)$ provides the required condition for solving the rest of the problems. Besides $\left(1\left(b\right)\right)$ will be used as an pointer function that helps us in identifying the places for which the perturbation is to be induced. Also the domain concerned over here helps one to look for edge factor defined by

$$E=\nabla\Omega\cdot\nabla\Omega \qquad \left(2\right)$$

Equation $\left(2\right)$ in turn helps us to define the cost function associated to E by

$$I_E\left(\Omega\right)=\int_\mu \nabla\Omega\cdot\nabla\Omega d\mu \qquad \left(3\right)$$

Solving $\left(3\right)$ we can obtain edges that happen in the medical image due to presence of the diseased condition of the organ. Topological Derivative along with the transformation techniques is an established tool in the field of medical image processing [1].Besides edge function as a function of perturbation parameter ε with restriction on ε helps in solving for edge in medical images [2]. Equation $\left(3\right)$ just forms an extension of these. With ε as an factor in the characteristic function helps to identify the ROI, in identifying the affected region in human organs in medical images. This is achieved with the help of Lebesgue measure [3].

In $\left(3\right)$ $\Omega\in W_p^k\left(\mu\right)$ corresponds to Sobolev space. Arriving the results in this paper for medical images depends on the relation among μ-the overall domain representation with main focus on ROI(Region of Interest) and its segmentation counterpart Ω, W_p^k - the sobolev space, Lebesgue Space- $L^p\left(\Omega\right)$ and the Weak derivative D^φ. Taking perturbed domain Ω_ε into consideration and solving for the boundary of Ω_ε by taking dirichlet and Neumann condition into account is attested with the help of $L^p\left(\Omega\right)$ [2].

W_p^k represents the category of space which are having uniformity in the functions belonging to $f_w\in L^p\left(\Omega\right)$ in such a way that the Weak derivative $D^\varphi f_w$ must exists in $L^p\left(\Omega\right)$ so that the index φ is conditioned to be $\left|\varphi\right|\le k$ .W_p^k naturally forms an important case for solving partial differential equation, which forms the main part of the problem here.Also the complex or real valued function f_ℓ which represents Lebesgue space in finite dimensions. Expression of f_ℓ in the integral form must be in p^{th} power which must be Lebesgue integrable. It's norm by definition will be

$$\|f_\ell\|_{L_p}=\left(\int_\Omega \left|f_\ell\right|^p d\mu\right)^{\frac{1}{p}} \qquad \left(4\right)$$

Considering generalized function and expressing $W_p^k\left(\mu\right)$ will define it as the space of functions f_w on μ domain in such a way it's partial derivatives $\partial_1^{i_1}f_w,\partial_2^{i_2}f_w,\partial_3^{i_3}f_w,\ldots\ldots\partial_n^{i_n}f_w$ for all multi indices $i=\left(i_1,i_2,i_{3,}\ldots\ldots,i_n\right)\in\mathbb{Z}_{\ge 0}^n$ with $i_1+i_2+i_3+\ldots\ldots+i_n\le k$, indicating the complex nature of data. Naturally, all of these are devised in such a way that it is present in $L^p\left(\Omega\right)$ space. Considering all these sobolev space in our problem is defined as follows,

$$W_p^k = W_p^k(\mu) \qquad 1 \le p \le \infty, \quad k \ge 1 \qquad (5)$$

Equation (5) forms an important formalizing part in our problem. With this the weak derivative is defined as

$$\int_\mu u D^\varphi \gamma = (-1)^{|\varphi|} \int_\mu v\gamma \qquad (6(a))$$

$$D^\varphi \gamma = \frac{\partial^{|\varphi|}\gamma}{\partial x_1^{\varphi_1}, \partial x_2^{\varphi_2}, \partial x_3^{\varphi_3}, \ldots\ldots \partial x_n^{\varphi_n}} \qquad (6(b))$$

Equation $(6(a))$ and $(6(b))$ forms the basic definition of Weak derivative which we will be used in the problem for finalizing the result. In $(6(a))$ and $(6(b))$ φ is multi-indexed. Weak derivative insists that the function should be continuous at the concerned point based on the Dini derivatives $D^+ f_w, D_+ f_w, D^- f_w \,\&\, D_- f_w$. Besides weak derivative is actually an integrable and also it lies in $L^p(\Omega)$ space. In $(6(a))$ and $(6(b))$ γ is such that $\gamma \in C_c^\infty(\mu)$ for all differentiable functions γ and is unique.

Sobolev space $W_p^k(\mu)$ has a norm given by,

$$\| f_w \|_{W_p^k(\mu)} = \sum_{i=0}^{k} \left(\sum_{\varphi:|\varphi|=i} \| D^\varphi f_w \|_{L^p(\Omega)} \right) \qquad (7)$$

from which we will be able to obtain $W^0{}_p{}^k(\mu)$ which is the closure of the space $C_{cl}^\infty(\mu)$ in the topology induced by this norm. Under the condition that μ is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $1 \le p \le \infty$, the localization property of $W_p^k(\mu)$ can be made use of in such a way that for all locally integrable functions $f_w : \mu \to \mathbb{R}$, we have

$$\partial^\varphi f_w \in L^p(\Omega) \quad for \quad 0 \le |\varphi| \le k \qquad (8)$$

then we have spaces formed under special cases, in such a way that

$$W^{k,2}(\mu) = H^k(\mu) \qquad (9)$$

II. TOPOLOGICAL DERIVATIVE

$D_T(\hat{x})$ has many application in science and engineering fields. Uniqueness about this $D_T(\hat{x})$ is that the inclusion of a perturbation and then calculating for the changes to bring out results. For optimization problems in the domain of medical images Lagrange multiplier addresses the issue [4]. Development of Algorithms which makes problem solver easier is achieved with the help of Lagrange multiplier.

One of the notable features in the application of $D_T(\hat{x})$ lies in the field of material science is in which Lagrange method also helps in achieving the result [5]. Application of Topological asymptotic expansion for restoration of gray scale images and image classification is well attested [6]. Creation of hole in the domain also leads to shape sensitivity. This concept was used in studying about the band gap in photonic crystal in condensed matter physics [7].

$D_T\left(\hat{x}\right)$ along with the level set method helps in selected the desired boundary of regions in the medical image. Not only the Level Set method can handle topological changes, it is also capable of removing holes when the object undergoing shape evolution. Algorithms developed based on these principles were proved to be effective for solving problems [8].

Existence of the perturbation is challenged by the topological asymptotic expansion for the improvement of the cost function is another interesting problem [9].Providing optimal solutions to problems that application of $D_T\left(\hat{x}\right)$ on the boundary and on the interior of the domain helps in achieving desired results [10]. Topological shape sensitivity in the case of Fluid dynamic is an nice application [11].

Shape reconstructions using $D_T\left(\hat{x}\right)$ along with level set method is an additional application of $D_T\left(\hat{x}\right)$ [12]. Applying diffusion coefficient to perturbation problem is already established [13].Besides consecutive works like segmentation of Carotid Artery with the help of different versions of $D_T\left(\hat{x}\right)$ is well attested .Speaking of diseased organs and the application of $D_T\left(\hat{x}\right)$ to the problem is already brought out by segmentation of Tumours [14]. Also $D_T\left(\hat{x}\right)$ is used for noise removal and edge detection too [15]. Using all of these concepts this paper had moved forwards a few steps for bringing out a detailed picture of diseased organs in medical images.

III. CALCULATION OF TOPOLOGICAL DERIVATIVE

Medical image under consideration must be treated in a matrix sense. Define domain μ & Ω to be a large matrix. μ matrix takes care of the medical image as a whole while the Ω matrix takes care of the segmentation part in the medical image. Another domain matrix Ω_ε takes care of the perturbed part in the segmented medical image. Initial stage of the problem is divided into $\mu \rightarrow$ Space related matrix, $\Omega \rightarrow$ Segmented matrix and $\Omega_\varepsilon \rightarrow$ Perturbed Matrix. This forms basic definition of the problem which will pave way for calculation of $D_T\left(\hat{x}\right)$. With this matrix in our hand the following paragraph shows the way that forms the other part of the calculations.

Further domain matrix Ω_ε calls for the shape derivative of the domain which is in turn expressed as

$$\left.\frac{d}{d\tau} I\left(\Omega_\tau\right)\right|_{\tau=0} = \lim_{\varepsilon\to 0} \frac{I\left(\Omega_\tau\right) - I\left(\Omega_\varepsilon\right)}{\tau} \qquad (10)$$

Interesting part of (10) is that for $\tau \in \mathbb{R}^+$ and τ to be small enough we will be getting still a finer detailed picture of the perturbed region. Here we will get another different matrix. This is called as $\Omega_\tau . \Omega_\varepsilon$, the perturbed matrix is different from that of Ω_τ, the shape derivative matrix because the latter gives the matrix representation of the image after undergoing changes that happened due to perturbation and also due other factors like τ . Medical image under consideration is defined as

$$\mu^d := \left\{\mu \cup \Omega^d \quad with\; d = 1,2,3,4,5,\ldots\ldots\ldots,(m\times n)\right\} \qquad (11)$$

Equation (11) gives us a basic representation of the medical image in a matrix sense. This starts the first step of our problem. Number of pixels for which the perturbation is to be introduced is selected from μ^d . Pixels thus selected are commonly named as $W_{\mu^p}\left(\Omega_\varepsilon\right)$. Since our problem speaks of diffusion taken to be perturbation, as a first step diffusion is introduced into the selected

$W_{\mu^p}(\Omega_\varepsilon)$. This is initial diffusion only. Tensor is created of zeros and ones which takes care of diffusion process. Handling the order of μ^d this particular tensor takes up the same order and is defined as $\mathrm{K}_I\left(W_{\mu^p}(\Omega_\varepsilon)\right)$. For further progress the initial conductivity of the problem must be changed.

The focus of the problem now lies in introducing a further diffusion ranging to certain depth. This becomes final diffusion. Along the same lines another tensor is created of ones and zeros. This time final diffusion is termed as $\mathrm{K}_F\left(W_{\mu^p}(\Omega_\varepsilon)\right)$. Starting from $\mu^d{}_{m\times n}$ and solving for the eigenvalues, eigenvectors and going to adjoint of $\mathrm{K}_F\left(W_{\mu^p}(\Omega_\varepsilon)\right)$. Name this adjoint the representation of which is given as

$$AjK_F = \begin{matrix} C_{K_{F11}} & C_{K_{F21}} & C_{K_{F31}} & C_{K_{F41}} & C_{K_{F51}} & \cdots & \cdots & \cdots & C_{K_{F(m-2),n1}} \\ C_{K_{F12}} & C_{K_{F22}} & C_{K_{F32}} & C_{K_{F42}} & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{K_{F13}} & C_{K_{F23}} & C_{K_{F33}} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{K_{F14}} & C_{K_{F24}} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{K_{F15}} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{K_{F1m}} & C_{K_{F2m}} & C_{K_{F3m}} & C_{K_{F4m}} & C_{K_{F5m}} & \vdots & \vdots & \vdots & C_{K_{F(m-2),(n-2)}} \end{matrix} \qquad (12)$$

With this and few additional steps one arrives at the desired solution for calculating the $D_T(\hat{x})$ calculated by using the gradient methods obtained among the pixels is expressed as

$$D_T\left(K_F{}^{m_i,n_j}\right) = \sum_{m_i,n_j} K_{F_\rho} \dddot{\Delta} m_{i_\rho} . \dddot{\Delta} \mathrm{n}_{j_\rho} - \sum_{m_i,n_j} K_F \dddot{\Delta} m_i . \dddot{\Delta} \mathrm{n}_j \qquad (13)$$

Where K_{F_ρ} represents the perturbation that happened due to the diffusion factor having a depth of ρ,

m_{i_ρ} & n_{j_ρ} represents pixels that had undergone perturbation to a depth ρ,

K_F represents the perturbation that happened due to the initial diffusion factored on itsy-bitsy note,

m_i & n_j represents pixels that had undergone perturbation on an initial stage.

$D_T\left(K_F{}^{m_i,n_j}\right)$ in (13) gives topological derivative as the variation functional of the cost function $c_f(\Omega_\varepsilon)$. This $c_f(\Omega_\varepsilon)$ provides an important solution by solving for the known parameters in the problems. This result still paves way for solving other spaces present within this problem and calculating for weak derivatives too.

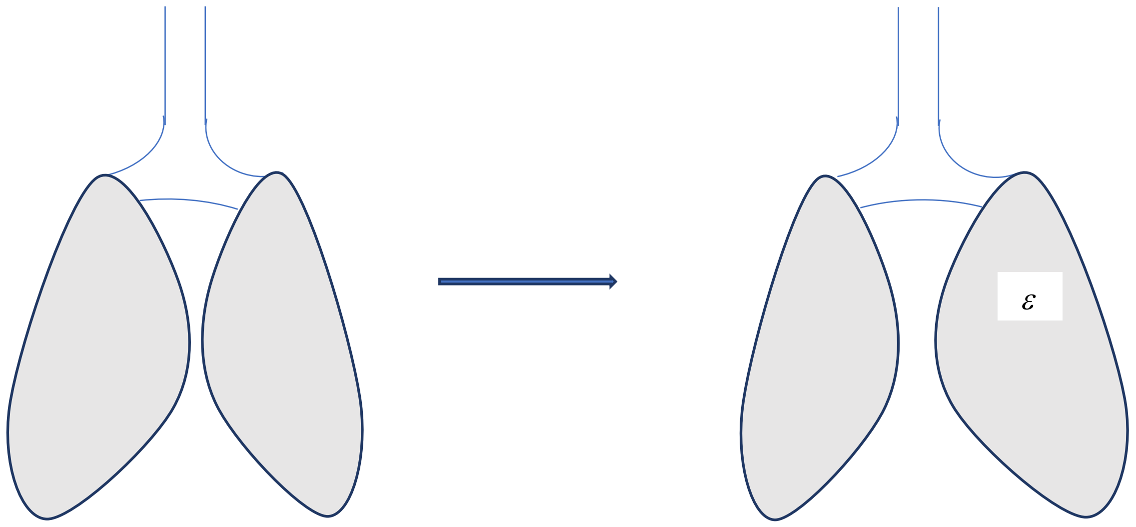

Figure 1. Illustration of Topological Derivative with the original domain Ω represent in the LHS and the RHS shows the affected domain Ω_ε . Above example shows the case of Lungs.

IV. METHODOLOGY

Treating μ^d as the input matrix which represent the form of medical image, the aim of this paper is to calculate weak derivative that helps to get edges defined on a note previously not possible. This represents the abnormalities present in the medical images, but still the consistence of continuity on the edges is maintained. This provides another advantage of the weak derivative.

Setting $\mathrm{M} = \mu^d{}_{m\times n}$ we will be able to calculate the eigenvalues, eigenvectors by treating this characteristic polynomial to zero. Then after the full diffusion process K is over one obtains the results for the cost function after going through the adjoint and it's related process to get the result of $D_T\left(\hat{x}\right)$.

$c_f\left(\Omega_\varepsilon\right)$ values and it's functional as a derivation, besides having a strong relation to $W_p^k\left(\mu\right)$ forms the next level of problem solving in the respective medical image. Here the associated spaces and it's variations are studied and marked which takes the medical image for the penultimate level of study.

By now the edges in the medical images have been focussed on a nice note. But to achieve a super resolution note on edges, segmented part, diseased organs….., weak derivative $D^\varphi\gamma$ comes into play. Property of $D^\varphi\gamma$ which makes it possible to have a value or maintain uniqueness in the regions of concentrated edges in the domain of medical images helps in achieving results not only in the edge part but also in the other region of medical images. This says $D^\varphi\gamma$ can assign values to other parts in image like segmented, borders, edges…., and so it in turns throws better results regarding μ^d , Ω_ε, $W_p^k\left(\mu\right)$ etc.,

V. RESULT

With results arrived in step wise format starting from topological derivative and ending with weak derivative one can arrive at results having detailed enhanced results regarding the edges in the medical images, besides concentrating on the medical image as a whole for further improvement in the regions of segmentation.

VI. CONCLUSION

This method in general gives result in a faster way than the previously available techniques. Plus, the advantage of the coding over here is that the processing of the images handling many types of data makes for both segmentation and at the same time addressing the usual problem of reducing the noise in the image in an efficient way. Obtaining an enhanced medical image is made possible with this technique. In the end comparing the processed image with the proficiency developed here with that of an ordinary image will show the diseased organ in an emphasized manner.

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