On certain energies of Fibonacci cubes

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Abstract

The eigenvalues of a graph *G* are the eigenvalues of its adjacency matrix. The energy of the graph is defined as the sum of the absolute values of all its eigenvalues. In this paper we compute different energies of Fibonacci cubes.

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1. Introduction

Let $G(V, E)$ be a simple connected undirected graph with p vertices and q edges. The adjacency matrix of G is the $0-1$ matrix $A = (A_{xy})$, where $A_{xy} = 1$ when there is an edge between vertices x and y in G and $A_{xy} = 0$, otherwise.

The characteristic polynomial of G is the characteristic polynomial of its adjacency matrix A and is denoted by $P_G(\lambda)$. The eigenvalues of G are the zeros of the characteristic polynomial and the *spectrum* of G is the multiset of eigenvalues of G denoted by $Spec(G)$. We write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}$$

where $\lambda_1, \lambda_2 \dots \lambda_p$ are the eigenvalues and m_i is the multiplicity of λ_i , $1 \leq i \leq p$. Unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \lambda_p$. The largest eigenvalue λ_1 is called the *index* of G. The terminology and definitions that we adopt are as in [5, 6, 7]. Since the adjacency matrix A is real and symmetric, all its eigenvalues are real. Further A has zero diagonal and consequently the trace of A is zero. If $\lambda_1, \lambda_2 \dots \lambda_p$ are the eigenvalues of G, then the *energy* of G is defined by

$$E(G) = \sum_{j=1}^{p} |\lambda_j|$$

The degree matrix of a graph G is a $p \times p$ diagonal matrix $D = (d_i)$, where d_i is the degree of the vertex v_i. The Laplace matrix L of G, is defined by $L = D - A$, where D is the degree matrix and A is the adjacency matrix of G [8]. The Laplace spectrum is the spectrum of the Laplace matrix.

Since L is real and symmetric, the Laplace spectrum is real. If $\mu_1, \mu_2 \ldots \mu_p$ are the eigenvalues ofthe Laplace matrix, then the Laplacian energy, denoted by $LE(G)$,is defined by

$$LE(G) = \sum_{i=1}^{p} \left|\mu_i - \frac{2q}{p}\right|$$

The matrix $Q = D + A$ is called the Q-Laplace matrix or the signless Laplace matrix of G. If $t_1 \geq t_2 \geq \ldots \geq t_p \geq 0$ are the eigenvalues of the Q-Laplace matrix of G then the Q-Laplacian energy of G is defined by

$$QE(G) = \sum_{i=1}^{p} \left|t_i - \frac{2q}{p}\right|$$

It is interesting to draw the attention of the mathematical community to rapidly growing applications of the theory of graph spectra. Besides classical and well documented applications to Chemistry and Physics, there are applications of graph eigenvalues in Computer Science in various investigations. There are also applications in several other fields like Biology, Geography, Economics and Social Sciences.

Graph spectra appear in internet technologies, pattern recognition, computer vision and in many other areas. One of the oldest applications (from 1970's) of graph eigenvalues in computer science is related to graphs called expanders. The recent progress on expander graphs and eigenvalues was initiated by problems in communication networks. Expanders can be constructed from graphs with a small second largest eigenvalue in modulus. Such class of graphs includes the so called Ramanujan graphs [5].

The energy of a graph was introduced by Ivan Gutman in 1978 [9]. However, the motivation for his definition appeared much earlier. In the 1930s the German scholar Erich Hückel put forward a method for finding approximate solutions of the Schrodinger equation of a class of organic molecules, the so-called "unsaturated conjugated hydrocarbons". Details of this approach often referred to as the "Hückel Molecular Orbital (HMO) theory" can be found in appropriate text books [3, 24, 26].

As late as 1956, Günthard and Primas realized that the matrix used in the Hückel method gave rise to a first-degree polynomial of the adjacency matrix of a certain graph related to the molecule being studied [12]. In 2006, Gutman and Zhou defined the Laplacian energy of a graph as the sum of the absolute deviations (i.e., distance from the mean) of the eigenvalues of its Laplacian matrix[8]. Remarkable variations of the concept of graph energy were developed for the incidence matrix [16], the signless Laplacian [25],the distance matrix [11].

The energy of cross products of some graphs [8], iterated line graphs of regular graphs [23], some self-complementary graphs [15], regular graphs [14] are obtained. Some of the works pertaining to the computation of $E(G)$ can be seen in [1, 9, 13].

2. Fibonacci cubes

Fibonacci cubes have been studied in several contexts, and the aim of their study goes beyond purely theoretical considerations. In computer science Fibonacci cubes are interesting from algorithmic point of view, since many algorithms that are known to be polynomial (or even linear) on the class of hypercubes work as well on Fibonacci cubes [4, 17].

Besides many appealing applications of Fibonacci cubes, these graphs have also been studied from a purely theoretical point of view. One of the reasons is due to their nice recursive structure and properties derived from it [18]. In particular several graph theoretical invariants are easily determined using their structural properties. It was proved in [4] that every Fibonacci cube has a Hamiltonian path, and in [21] the independence number of Fibonacci cubes is determined. Various enumerative sequences of these graphs have been determined as well: number of vertices of a given degree [19], number of vertices of a given eccentricity [2] and number of isometric subgraphs isomorphic to Q_k [20]. The counting polynomial of this last sequence is known as cubic polynomial and has very nice properties. In [22] the author gave the number of maximal hypercubes of dimension k in Γ_n (that is, induced subgraphs H of Γ_n isomorphic to Q_k, such that there exists no induced subgraph H' isomorphic to Q_{k+1}, with $H \subseteq H'$) and determined the exact number of maximal hypercubes for every k and n.

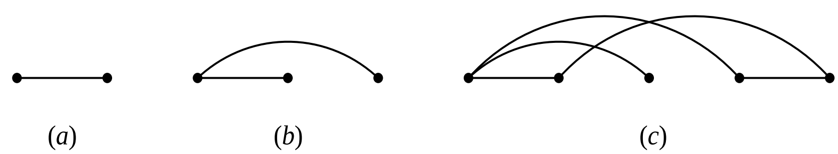

Figure 1: (*a*) Γ_1 (*b*) Γ_2 (*c*) Γ_3

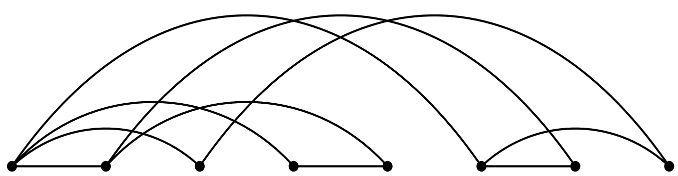

Figure 2: Fibonacci cube Γ_4

The Fibonacci numbers are defined as $F_0 = 0$, $F_1 = 1$, $F_n=F_{n-1}+ F_{n-2}$, for $n > 1$. A Fibonacci string is a binary string with no consecutive 1's. The vertex set of Fibonacci cube Γ_n, $n \geq 0$, contains Fibonacci strings of length n, two vertices are adjacent if they differ exactly in one bit. In other words, Γ_n is a graph obtained from hypercube Q_n by removing all vertices that contain at least two consecutive 1's. Obviously, Fibonacci cube of order n is a subgraph of the hypercube Q_n. Note that $\Gamma_1 = K_2$ and Γ_2 is a path on three vertices.

The number of vertices of an n-dimensional Fibonacci cube Γ_n is F_{n+2}. A Fibonacci cube of order n can be decomposed into two Fibonacci cubes Γ_{n-1} and Γ_{n-2} which are connected by F_n edges. The number of edges of Γ_n is $F_{n+1} + \sum_{i=1}^{n-2} F_i\, F_{n+1-i}$ and the diameter of Γ_n is n.

As there is no easy way to find the eigenvalues of a graph G, the computation of the actual value of $E(G)$ is an interesting problem in graph theory. In this paper we obtain different spectra of Fibonacci cubes network using MATLAB.

3. Calculating the energy

The following MATLAB program generates the adjacency matrix of Fibonacci cube Γ_n and also calculates its energy.

```
function C=fib(k,A2,A1)
x1=0;
```

```
x2=1;
for i=1:k
x=x1+x2;
x1=x2;
x2=x;
end
C=blkdiag(A2,A1);
for i=1:x1
C(i,x+i)=1; C(x+i,i)=1:
end

function B=fibspectrum(n)
x=0;
A1=[0 1; 1 0];
A2=[0 1 1; 1 0 0; 1 0 0];
switch n
case 1
C=A1;
case 2
C=A2;
otherwise
for i=3:n
C=fib(i,A2,A1);
A1=A2;
A2=C;
end
end

disp(Adjacency Matrix);
C
Spectrum=eig(C)
Energy=sum(abs(spectrum))
end
```

Proof of correctness

The function ‘blkdiag’ in the function fib, concatenates the matrices A2 and A1. The second ‘for loop’ in the function fib generates the identity matrix required in the adjacency matrix of the Fibonacci graph in every dimension. The ‘for loop’ in the function fibspectrum runs the function fib $n - 2$ times to generate the adjacency matrix of Fibonacci graph.

4. Calculating the Laplacian energy

The following **MATLAB** program generates the Laplace matrix of Fibonacci cube Γ_n and also calculates its Laplacian energy.

```
function B=fibspectrum(n)
x=0;
A1=[0 1; 1 0];
A2=[0 1 1; 1 0 0; 1 0 0];
switch n
case 1
C=A1;
case 2
C=A2;
Otherwise
for i=3:n
C=fib(i,A2,A1);
A1=A2;
A2=C;
end
end

disp(Laplace Matrix);
D=diag(sum(C));
L=D–C
Laplace Spectrum=eig(L)
Laplacian Energy=sum(abs(Laplace Spectrum-n/nnz(C)))
end
```

5. Calculating the Q-Laplacian energy

The following MATLAB program generates the Q-Laplace matrix of Fibonacci cube Γ_n and also calculates its Q-Laplacian energy.

```
function B=fibspectrum(n)
x=0;
A1=[0 1; 1 0];
A2=[0 1 1; 1 0 0; 1 0 0];
switch n
case 1
C=A1;
case 2
C=A2;
otherwise
for i=3:n
C=fib(i,A2,A1);
A1=A2;
A2=C;
end
end
```

```
disp(Laplace Matrix);
D=diag(sum(C));
Q=D+C
Q-Laplace Spectrum=eig(Q)
Q-Laplacian Energy=sum(abs(Q-Laplace Spectrum-n/nnz(C)))
end
```

6. Conclusion

In this paper we have computed different energies of Fibonacci cubes. We focus our attention in identifying computer programs or suitable software to compute different energies of certain interconnection networks such as cube connected cycles and enhanced hypercubes.

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