An M/M/2 Queue with heterogeneous servers, system disaster, server repair, customers' impatience, balking and reneging

R. Sudhesh[1], P. Balakrishnan[2] and P. Savitha[3]
[1]Department of Mathematics
Bharathidasan Institute of Technology (BIT) Campus
Anna University, Tiruchirappalli-620024, India
[2]Department of Mathematics
Adhi Parasakkthi Engineering College, Melmaruvathur, India
[3]Department of Mathematics
SASTRA Deemed University, Thanjavur 613402, India
E-mail: sudheshanna@gmail.com; balakrishnamaths@gmail.com; savitha@aucev.edu.in

Abstract

This paper analyses a two heterogeneous servers queueing system with disastrous breakdowns, server repair, balking, reneging and impatience of customers. The customers become impatient when the system is down. After the occurrence of the disaster, a repair process starts immediately and the server start doing service of the arrival new customers when the servers get repaired. A steady state system size probabilities of number customers present in the system are obtained in closed form using Gauss hypergeometric functions and incomplete beta functions.

Keywords: Heterogeneous servers - system disaster - server repair - impatient customers - balking and reneging

Mathematics Subject Classification 60K25

1 Introduction

Multi-server queueing systems with heterogeneous servers are useful for analyzing a wide variety of real systems, including manufacturing systems, service systems, telecommunications and computer systems. Multi-server queueing systems with several heterogeneous servers are useful for analysing a wide variety of real systems, including adequate modelling of the system data transmission nodes and in the design of asynchronous transfer mode network; see for instance [3, 6, 9].

Heterogeneous servers are the servers who have varying individual service rates based on their efficiency of providing service. Many of the researchers concentrate mutiserver queueing model with homogeneous servers i.e., service rates of all the servers are constant. But in real life situations, the service rates of servers are heterogeneous in nature. Service stations which are not mechanically controlled like checkout counters, grocery stores, banks and departments etc. have heterogeneous service because one cannot expect human servers to work at constant rate. It is essential that one can analyse the multiserver queueing system with heterogeneous servers.

Singh [12] extends the work of Krishnamoorthy [4] on two heterogeneous servers by involving balking and revealed that the heterogeneous system is better than the corresponding homogeneous system. Krishnamoorthy and Sreenivasan [10] discussed an M/M/2 heterogeneous servers queueing model where one server remains idle but the other goes on working vacation in the absence of waiting customers.

Balking is a widespread effect of not joining a queue because the arriving customer estimates the queue to be too long. The concept of balking finds its applications not only in daily life, but in computer communication, production systems and hospital management

Queueing with customer impatience finds its applications in various areas like call centers, packet-switched communication networks, hospitals, perishable inventory systems etc. Queueing systems with customer impatience can be treated for relating some types of perishable inventory systems as there is an correspondence between queueing systems with reneging and perishable inventory systems.

The on hand inventory can be regarded as a queue, the demand completion as completion of service, the products pending in the form of replenishment as arrivals to the queueing system, and the life time of a product as the impatience (reneging) time. Customer reneging and product perishing are analogous event. A customer whose patience time expires leaves the queue whereas a product made to stock whose lifetime expires is removed from the inventory.

In supply chains, the perishable items like vegetables, fruits etc. in the congestion situations become worthless if they are not reached to the vendors (customers) at appropriate time as they may get damaged i.e. the perishable items can be modeled as the reneged customers.

In the call centres, a calling customer generally hangs up before service agent and thus gets reneged. In packet switched communication networks with time critical traffic, a packet loses its value if it is not transmitted within a given time interval. The patients (customers) who leave the emergency rooms in hospitals without been observed are also regarded as reneged customers. Kidney transplant waiting system can be considered as a queue with reneging, where reneging occurs because a customer that is waiting for a kidney may die.

Rakesh kumar and Sharma [11] considered a multi-server, finite capacity queueing model with discouraged arrivals, reneging and retention of reneged customers and they derived the steady-state solution of the model, and also discussed cost-profit analysis. Vijaya Laxmi and Jyothsna [15] studied a two server queue with balking, reneging and working vacations and obtained the steady-state distribution of the number of customers in the system at pre-arrival and arbitrary epochs.

But Recently in Amina Angelika Bouchentouf et al. [2], consider a heterogeneous two-server queueing model with Bernoulli feedback, reneged customers, retention of reneged customers and no waiting line. In that model, customers being receiving service may disengage before service completion and using certain mechanism they can be retained with some probability. Using a recursive method for obtained the stationary state probabilities and then deduced useful performance measures.

Recently Henry Baumann et al., (2018) in [8] discussed about the number of overtakes a stationary customer in an M/M/2 queue by using generating function. Also find the expectation and the variance. In this case, even the FCFS discipline allows overtaking customers with a relatively long service time may be over taken by customers with relatively short service time served by another server.

Yang et al.[14] considered a single waiting line queueing system of two constant rates servers but two types of customers i.e., high-type customer with more delay-sensitive and brings less workload to the system than the low-type customer, and they discussed the equilibrium queueing strategies of two types of customers in the two-server queue.

Sudhesh et al. [13] studied transient and steady state analysis of a two heterogeneous servers queue with system disaster where customers become impatient while the system is down. We extend this work and considered a server repair, balking, reneging and impatience of customers. The steady state system size probabilities are obtained in closed form of such queueing system using hypergeometric functions and incomplete beta functions.

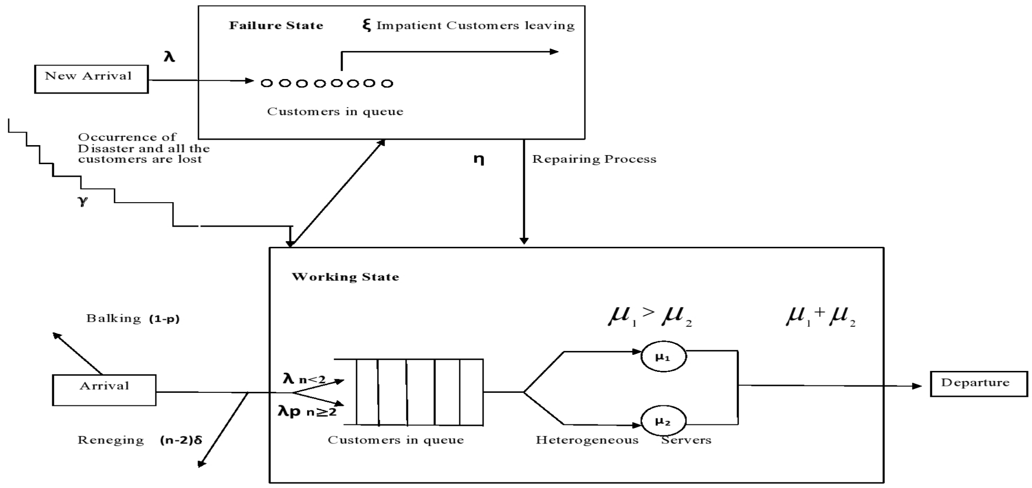

Figure 1: Diagrammatic Representation of the Model

2 Model Description

We consider a two-server heterogeneous queueing system with disastrous breakdowns,repair, balking and reneging and customer impatience where customers became impatient when the system is down. Customers arrive according to a Poisson Process with rate λ. Service is exponentially distributed where the two servers provide heterogeneous service with different service rates μ_1 and μ_2 such that $\mu_1 > \mu_2$.Each customer needs only one server for service and the customers select the servers on fastest server first (FSF) basis.

A customer who on arrival finds at least two customers in the system, either decides to enter the queue with probability p or balk with probability $1-p$. Let $\lambda_p = \lambda p$. After joining the queue, each customer will wait a certain length of time T for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time T is assumed to be distributed according to an exponential distribution with mean $1/\delta$. Since the arrival and the departure of the impatient customers without service are independent, the reneging rate when there are n customers is $(n-2)\delta$.

When the system is idle or busy, disaster occurs according to a Poisson process of rate γ. Whenever a disaster occurs at the system, all present customers (waiting and served) are flushed out from the system and both the servers become inactivated. A repair process then starts immediately and the repair time of the system is exponentially distributed with mean η^{-1}. When the system is down, inoperative, and undergoing a repair process,new arrivals become impatient. Each individual customer, upon arrival, activates a random-duration impatience timer with parameter ξ. If the timer expires before the system is repaired, the customer abandons the queue and never return.

Let $\{(X(t), Y(t)), t \geq 0\}$ be a two-dimensional continuous-time Markov chain, where $X(t)$ denotes the number of customers in the system at time t and $Y(t)$ represents the state of the system at time t with state space $S = \{(n, j) : n = 0, 1, 2, ...j = 0, 1, 2, 3, 4\}$. The state transition diagram of the system is given in figure 1. If $Y(t) = 3$, the system is functioning and both the servers are serving customers, whereas if $Y(t) = 4$, the system is down and undergoing a repair process.The pictorial representation of the model is shown in Fig.1.

The state $(0,0)$ represents that the system is empty and servers are in 'ON' state, $(1,1)$ represents that there is one customer in the system and served by faster server and $(1,2)$ represents that there is one customer in the system and served by slower server. The state $(n,3)$, $n = 2,3,4,\dots$, represents that there are n customers in the system when the system is in working state and the state $(n,4)$, $n = 0,1,2,\dots$, represents that there are n customers in the system when the system is in failure state.

Let $P_{nj}(t)$ denote the time dependent system size probabilities where there are n customers in the system at time t and j takes values $0,1,2,3$ and 4. Mathematically,

$$P_{nj}(t) = P[X(t) = n, Y(t) = j], n = 0,1,2,\dots; j = 0,1,2,3,4.$$

2.1 Governing Equations

With the underlying assumptions, the behaviour of resulting system is described by a set of Chapman Kolmogorov forward equations which can be written as:

$$P'_{0,0}(t) = -(\lambda+\gamma)P_{0,0}(t) + \mu_1 P_{1,1}(t) + \mu_2 P_{1,2}(t) + \eta P_{0,4}(t) \quad (2.1)$$

$$P'_{1,1}(t) = -(\lambda+\mu_1+\gamma)P_{1,1}(t) + \lambda P_{0,0}(t) + \mu_2 P_{2,3}(t) + \eta P_{1,4}(t) \quad (2.2)$$

$$P'_{1,2}(t) = -(\lambda+\mu_2+\gamma)P_{1,2}(t) + \mu_1 P_{2,3}(t) \quad (2.3)$$

$$P'_{2,3}(t) = -(\lambda_p+\mu+\gamma)P_{2,3}(t) + \lambda P_{1,1}(t) + \lambda P_{1,2}(t) + (\mu+\delta)P_{3,3}(t) + \eta P_{2,4}(t) \quad (2.4)$$

$$P'_{n,3}(t) = -(\lambda_p+\mu+(n-2)\delta+\gamma)P_{n,3}(t) + \lambda_p P_{(n-1),3}(t) + (\mu+(n-1)\delta)P_{(n+1),3}(t) + \eta P_{n,4}(t), \quad n \geq 3 \quad (2.5)$$

$$P'_{0,4}(t) = -(\lambda+\eta)P_{0,4}(t) + \xi P_{1,4}(t) + \gamma\left[1 - \sum_{n=0}^{\infty} P_{n,4}(t)\right] \quad (2.6)$$

$$P'_{n,4}(t) = -(\lambda+\eta+n\xi)P_{n,4}(t) + \lambda P_{(n-1,)4}(t) + (n+1)\xi P_{(n+1),4}(t), \quad n \geq 1 \quad (2.7)$$

where $\mu = \mu_1 + \mu_2$.

We assume that the number of customers present initially is random with the probability $p_r, r = 0,1,2,\cdots$ where r represent the number of customers in the system initially.

3 Steady-state Probabilities

In this section, the steady-state probabilities are derived by replacing zero to the left-hand side of the system of equations (2.1) - (2.7). From these equations we obtain,

$$P_{0,0} = \left[\frac{(a+c+\mu_1)\mu_1\mu_2}{a^2c + c\gamma\mu_1}\right]P_{2,3} + \left[\frac{\eta(a+\mu_1)}{a^2+\gamma\mu_1}\right]P_{0,4} + \left[\frac{\eta\mu_1}{a^2+\gamma\mu_1}\right]P_{1,4}, \quad (3.1)$$

$$P_{1,1} = \left[\frac{\lambda\mu_1\mu_2 + ac\mu_2}{a^2c + c\gamma\mu_1}\right]P_{2,3} + \left[\frac{\lambda\eta}{a^2+\gamma\mu_1}\right]P_{0,4} + \left[\frac{a\eta}{a^2+\gamma\mu_1}\right]P_{1,4}, \quad (3.2)$$

$$P_{1,2} = \left[\frac{\mu_1}{\lambda+\gamma+\mu_2}\right]P_{2,3}, \quad (3.3)$$

$$P_{2,3} = \left[\frac{\lambda^2\eta c}{X}\right]P_{0,4} + \left[\frac{\lambda\eta ac}{X}\right]P_{1,4} + \left[\frac{\eta c(a^2+\gamma\mu_1)}{X}\right]P_{2,4} + \left[\frac{c(\mu+\delta)(a^2+\gamma\mu_1)}{X}\right]P_{3,3}, \quad (3.4)$$

where

$$\begin{aligned} a =& \lambda + \gamma, \\ c =& \lambda + \gamma + \mu_2, \quad \text{and} \\ X =& (a^2 + \gamma\mu_1)cd - \lambda\mu_2(ac + \lambda\mu_1) - \mu_1(a^2 + \gamma\mu_1). \end{aligned}$$

Using the identity from Lorentzen and Waadeland [8], we obtain for $n \geq 1$,

$$P_{n,4} = \psi_n P_{0,4}, \tag{3.5}$$

where

$$\psi_n = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}(\frac{\eta}{\xi} + i)} \frac{{}_1F_1(n+1; \frac{\eta}{\xi} + n + 1; -\frac{\lambda}{\xi})}{{}_1F_1(1; (\frac{\eta}{\xi} + 1); -\frac{\lambda}{\xi})}$$

and

$$P_{0,4} = \gamma[\lambda + \eta + \gamma - \xi\psi_1 + \gamma \sum_{n=1}^{\infty} \psi_n]^{-1}. \tag{3.6}$$

To find $P_{n,3}, n \geq 3$, define the probability generating function

$$G(z) = \sum_{n=1}^{\infty} P_{(n+2),3} z^n \tag{3.7}$$

and

$$G^{'}(z) = \sum_{n=1}^{\infty} n P_{(n+2),3} z^{n-1}.$$

Using Eq.(2.5), we yield the differential equation

$$G^{'}(z) - \Big[\frac{\lambda_p z - \mu}{z\delta} + \frac{\gamma}{\delta(1-z)}\Big] G(z) - \frac{\mu + \delta}{\delta(1-z)} P_{3,3} + \frac{\lambda_p z}{\delta(1-z)} P_{2,3} + \frac{\eta}{\delta(1-z)} \sum_{n=1}^{\infty} P_{(n+2),4} z^n = 0. \tag{3.8}$$

To solve the above first order linear differential equation, an integrating factor can be found as

$$I.F = e^{-\int(\frac{\lambda_p}{\delta} - \frac{\mu}{z\delta} + \frac{\gamma}{\delta(1-z)})dz} = e^{-\frac{\lambda_p z}{\delta}} z^{\frac{\mu}{\delta}} (1-z)^{\frac{\gamma}{\delta}}$$

The general solution to the differential equation is

$$G(z) = z^{-\frac{\mu}{\delta}} (1-z)^{-\frac{\gamma}{\delta}} \Big[\Big(\frac{\mu}{\delta} + 1\Big) P_{3,3} A(z) - \frac{\lambda_p}{\delta} P_{2,3} B(z) - \frac{\eta}{\delta} \sum_{n=1}^{\infty} P_{(n+2),4} C(z)\Big], \tag{3.9}$$

where

$$A(z) = \int_0^z e^{\frac{\lambda_p}{\delta}(z-\chi)} \chi^{\frac{\mu}{\delta}} (1-\chi)^{\frac{\gamma}{\delta}-1} d\chi, \tag{3.10}$$

$$B(z) = \int_0^z e^{\frac{\lambda_p}{\delta}(z-\chi)} \chi^{\frac{\mu}{\delta}+1} (1-\chi)^{\frac{\gamma}{\delta}-1} d\chi \quad \text{and} \tag{3.11}$$

$$C(z) = \int_0^z e^{\frac{\lambda_p}{\delta}(z-\chi)} \chi^{\frac{\mu}{\delta}+n} (1-\chi)^{\frac{\gamma}{\delta}-1} d\chi. \tag{3.12}$$

To obtain $P_{3,3}$ in terms of $P_{2,3}$, let us determine (3.10)-(3.12) for limit z tending to 1. Using the identity from Abramowitz and Stegun [1], for $Re(u) > 0, Re(v) > 0$,

$$\int_0^w \chi^{v-1}(w-\chi)^{u-1}e^{\beta\chi}d\chi = B(u,v)w^{u+v-1} \quad {}_1F_1(v; u+v; \beta w), \tag{3.13}$$

with

$$\begin{aligned} B(b,c) &= \int_0^1 t^{(b-1)}(1-t)^{(c-1)}dt, b>0, c>0 \quad \text{the Beta function;} \\ {}_1F_1(\alpha;\beta;z) &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!} \quad \text{the degenerate hypergeometric function and} \\ (\alpha)_k &= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \quad \text{the Pochhammer symbol.} \end{aligned}$$

Substituting $z = 1, 1-\chi = t$ and using identity (3.13), in Eqs. (3.10) - (3.12), we obtain

$$\begin{aligned} A(1) =& \int_0^1 e^{\frac{\lambda_p}{\delta}t}(1-t)^{\frac{\mu}{\delta}}t^{(\frac{\gamma}{\delta}-1)}dt, \\ =& B(\frac{\gamma}{\delta}, \frac{\mu}{\delta}+1) \quad {}_1F_1\Big(\frac{\mu}{\delta}+1; \frac{\gamma}{\delta}+\frac{\mu}{\delta}+1; \frac{\lambda_p}{\delta}\Big), \\ A(1) =& K\Big(\frac{\lambda_p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta}+1\Big). \end{aligned} \tag{3.14}$$

$$\begin{aligned} B(1) =& \int_0^1 e^{\frac{\lambda_p}{\delta}t}(1-t)^{\frac{\mu}{\delta}+1}t^{(\frac{\gamma}{\delta}-1)}dt, \\ =& B(\frac{\gamma}{\delta}, \frac{\mu}{\delta}+2) \quad {}_1F_1\Big(\frac{\mu}{\delta}+2; \frac{\gamma}{\delta}+\frac{\mu}{\delta}+2; \frac{\lambda_p}{\delta}\Big), \\ B(1) =& K\Big(\frac{\lambda_p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta}+2\Big). \end{aligned} \tag{3.15}$$

$$\begin{aligned} C(1) =& \int_0^1 e^{\frac{\lambda_p}{\delta}t}(1-t)^{\frac{\mu}{\delta}+n}t^{(\frac{\gamma}{\delta}-1)}dt, \\ =& B(\frac{\gamma}{\delta}, \frac{\mu}{\delta}+n+1) \quad {}_1F_1\Big(\frac{\mu}{\delta}+n+1; \frac{\gamma}{\delta}+\frac{\mu}{\delta}+n+1; \frac{\lambda_p}{\delta}\Big), \\ C(1) =& K\Big(\frac{\lambda_p}{\delta}, \frac{\gamma}{\delta}, \frac{\mu}{\delta}+n+1\Big). \end{aligned} \tag{3.16}$$

where $K(a,b,c) = B(b,c) \quad {}_1F_1(c; b+c; a)$.
Now determining $G(z)$ for limit z tending to 1 yields,

$$\begin{aligned} \lim_{z\to 1} G(z) =& G(1) \\ =& \Big[(\frac{\mu}{\delta}+1)P_{3,3}A(1) - \frac{\lambda_p}{\delta}P_{2,3}B(1) - \frac{\eta}{\delta}\sum_{n=1}^{\infty}P_{(n+2),4}C(1)\Big] \times \lim_{z\to 1}(1-z)^{-\frac{\gamma}{\delta}}. \end{aligned}$$

Since $0 \leq G(1) = \sum_{n=1}^{\infty} P_{(n+2),3} \leq 1$ and $\lim_{z\to 1}(1-z)^{-\frac{\gamma}{\delta}} = \infty$, we must have the term

$$(\frac{\mu}{\delta}+1)P_{3,3}A(1) - \frac{\lambda_p}{\delta}P_{2,3}B(1) - \frac{\eta}{\delta}\sum_{n=1}^{\infty}P_{(n+2),4}C(1) = 0$$

and this condition gives

$$P_{3,3}=\frac{\lambda_p}{\mu+\delta}\frac{B(1)}{A(1)}P_{2,3}+\frac{\eta}{\mu+\delta}\frac{C(1)}{A(1)}\sum_{n=1}^{\infty}P_{(n+2),4}. \tag{3.17}$$

Using the above relation in (3.9), we obtain

$$G(z) \;=\; z^{-\frac{\mu}{\delta}}(1-z)^{-\frac{\gamma}{\delta}}\Big[\frac{\lambda_p}{\delta}P_{2,3}\Big(\frac{B(1)}{A(1)}A(z)-B(z)\Big)+\frac{\eta}{\delta}\sum_{n=1}^{\infty}P_{(n+2),4}\Big(\frac{C(1)}{A(1)}A(z)-C(z)\Big)\Big]. \tag{3.18}$$

To have $G(z)$ as a series, we expand the integrals $A(z), B(z)$ and $C(z)$ in series. The function $A(z)$ can be written in terms of incomplete beta function

$$B(x;a,b) \;=\; \int_0^x t^{(a-1)}(1-t)^{(b-1)}dt$$

and expressing the incomplete Beta function in terms of Gauss hypergeometric function and expanding the hypergeometric function in series gives

$$B(x;a,b)=\frac{x^a(1-x)^b}{a}F(a+b,1;a+1;x),$$
$$F(a;b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k.$$

Eq.(3.10) becomes,

$$\begin{aligned}
A(z) =& e^{\frac{\lambda_p}{\delta}z}\int_0^z\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n\chi^{\frac{\mu}{\delta}+n}(1-\chi)^{\frac{\gamma}{\delta}-1}d\chi,\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n B(z;\frac{\mu}{\delta}+n+1;\frac{\gamma}{\delta}),\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n\frac{z^{\frac{\mu}{\delta}+n+1}(1-z)^{\frac{\gamma}{\delta}}}{\frac{\mu}{\delta}+n+1}F\Big(n+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+1,1;\frac{\mu}{\delta}+n+2;z\Big),\\
A(z) =& e^{\frac{\lambda_p}{\delta}z}(1-z)^{\frac{\gamma}{\delta}}z^{\frac{\mu}{\delta}}\sum_{n=0}^{\infty}H(n)z^{n+1},
\end{aligned} \tag{3.19}$$

where

$$H(n)=\sum_{k=0}^{n}\frac{(-1)^{n-k}}{\Big(\frac{\mu}{\delta}+n-k+1\Big)(n-k)!}\frac{\Big(n-k+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+1\Big)_k}{\Big(n-k+\frac{\mu}{\delta}+2\Big)_k}\quad\Big(\frac{\lambda_p}{\delta}\Big)^{n-k}.$$

Similarly, Eq.(3.11) becomes,

$$\begin{aligned}
B(z) =& e^{\frac{\lambda_p}{\delta}z}\int_0^z \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n \chi^{n+\frac{\mu}{\delta}+1}(1-\chi)^{\frac{\gamma}{\delta}-1}d\chi,\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n B(z;n+\frac{\mu}{\delta}+2;\frac{\gamma}{\delta}),\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n \frac{z^{n+\frac{\mu}{\delta}+2}(1-z)^{\frac{\gamma}{\delta}}}{n+\frac{\mu}{\delta}+2}F\Big(n+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+2,1;n+\frac{\mu}{\delta}+3;z\Big),\\
B(z) =& e^{\frac{\lambda_p}{\delta}z}(1-z)^{\frac{\gamma}{\delta}}z^{\frac{\mu}{\delta}}\sum_{n=0}^{\infty}J(n)z^{n+2}, \qquad (3.20)
\end{aligned}$$

where

$$J(n) = \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\Big(\frac{\mu}{\delta}+n-k+2\Big)(n-k)!}\frac{\Big(n-k+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+2\Big)_k}{\Big(n-k+\frac{\mu}{\delta}+3\Big)_k}\quad\Big(\frac{\lambda_p}{\delta}\Big)^{n-k}$$

and Eq.(3.12) becomes,

$$\begin{aligned}
C(z) =& e^{\frac{\lambda_p}{\delta}z}\int_0^z \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n \chi^{2n+\frac{\mu}{\delta}}(1-\chi)^{\frac{\gamma}{\delta}-1}d\chi,\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n B(z;2n+\frac{\mu}{\delta}+1;\frac{\gamma}{\delta}),\\
=& e^{\frac{\lambda_p}{\delta}z}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\lambda_p}{\delta}\Big)^n \frac{z^{2n+\frac{\mu}{\delta}+1}(1-z)^{\frac{\gamma}{\delta}}}{2n+\frac{\mu}{\delta}+2}F\Big(2n+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+1,1;2n+\frac{\mu}{\delta}+2;z\Big),\\
C(z) =& e^{\frac{\lambda_p}{\delta}z}(1-z)^{\frac{\gamma}{\delta}}z^{\frac{\mu}{\delta}}\sum_{n=0}^{\infty}K(n)z^{2n+1}, \qquad (3.21)
\end{aligned}$$

where

$$K(n) = \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\Big(\frac{\mu}{\delta}+2n-2k+2\Big)(n-k)!}\frac{\Big(2n-2k+\frac{\mu}{\delta}+\frac{\gamma}{\delta}+1\Big)_k}{\Big(2n-2k+\frac{\mu}{\delta}+2\Big)_k}\quad\Big(\frac{\lambda_p}{\delta}\Big)^{n-k}z^{-k}.$$

Using (3.19), (3.20) and (3.21) in (3.18) and expanding the exponential terms e^{az} in series, we obtain

$$\begin{aligned}
G(z) =& \frac{\lambda_p}{\delta}P_{2,3}\Bigg[\frac{B(1)}{A(1)}\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\lambda_p}{\delta}\Big)^n\frac{1}{n!}z^n\Bigg)\Bigg(\sum_{n=0}^{\infty}H(n)z^{n+1}\Bigg)-\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\lambda_p}{\delta}\Big)^n\frac{1}{n!}z^n\Bigg)\Bigg(\sum_{n=0}^{\infty}J(n)z^{n+2}\Bigg)\Bigg]\\
&+\frac{\eta}{\delta}\sum_{k=1}^{\infty}P_{(k+2),4}\Bigg[\frac{C(1)}{A(1)}\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\lambda_p}{\delta}\Big)^n\frac{1}{n!}z^n\Bigg)\Bigg(\sum_{n=0}^{\infty}H(n)z^{n+1}\Bigg)-\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\lambda_p}{\delta}\Big)^n\frac{1}{n!}z^n\Bigg)\Bigg(\sum_{n=0}^{\infty}K(n)z^{2n+1}\Bigg)\Bigg] \qquad (3.22)
\end{aligned}$$

Using the Cauchy product,

$$\left(\sum_{n=0}^{\infty} a_n z^{n+1}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty} z^n \sum_{r=1}^{n} a_{r-1} b_{n-r},$$

$$\left(\sum_{n=0}^{\infty} a_n z^{n+2}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=2}^{\infty} z^n \sum_{r=2}^{n} a_{r-2} b_{n-r} \quad \text{and}$$

$$\left(\sum_{n=0}^{\infty} a_n z^{2n+1}\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty} z^n \sum_{r=0}^{[\frac{n-1}{2}]} a_r b_{n-2r-1}$$

we yield,

$$G(z) = \Bigg(\Big[\sum_{n=1}^{\infty}\sum_{r=1}^{n}\Big(\frac{\lambda_p}{\delta}\Big)^{n-r}\frac{1}{(n-r)!}H(r-1)\Big] \times \Big[\frac{\lambda_p}{\delta}P_{2,3}\frac{B(1)}{A(1)} + \frac{\eta}{\delta}\sum_{k=1}^{\infty}P_{(k+2),4}\frac{C(1)}{A(1)}\Big] - \frac{\lambda_p}{\delta}P_{2,3}$$
$$\sum_{n=2}^{\infty}\sum_{r=2}^{n}\Big(\frac{\lambda_p}{\delta}\Big)^{n-r}\frac{1}{(n-r)!}J(r-2) - \frac{\eta}{\delta}\sum_{k=1}^{\infty}P_{(k+2),4}\sum_{n=1}^{\infty}\sum_{r=0}^{[\frac{n-1}{2}]}K(r)\Big(\frac{\lambda_p}{\delta}\Big)^{n-2r-1}\frac{1}{(n-2r-1)!}\Bigg)z^n \tag{3.23}$$

and this is the solution to the differential equation Eq. (3.8). The PGF $G(z)$ gives the probabilities $P_{n,3}, n \geq 3$ in terms of $P_{2,3}$ and $P_{n,4}, n \geq 3$ as

$$P_{n,3} = \Big[\sum_{r=1}^{n}\Big(\frac{\lambda_p}{\delta}\Big)^{n-r}\frac{1}{(n-r)!}H(r-1)\Big] \times \Big[\frac{\lambda_p}{\delta}P_{2,3}\frac{B(1)}{A(1)} + \frac{\eta}{\delta}\sum_{k=1}^{\infty}P_{(k+2),4}\frac{C(1)}{A(1)}\Big] - \frac{\lambda_p}{\delta}P_{2,3}$$
$$\sum_{r=2}^{n}\Big(\frac{\lambda_p}{\delta}\Big)^{n-r}\frac{1}{(n-r)!}J(r-2) - \frac{\eta}{\delta}\sum_{k=1}^{\infty}P_{(k+2),4}\sum_{r=0}^{[\frac{n-1}{2}]}K(r)\Big(\frac{\lambda_p}{\delta}\Big)^{n-2r-1}\frac{1}{(n-2r-1)!} \quad n \geq 3. \tag{3.24}$$

Thus Eqs. (3.1) - (3.6) and (3.24) completely determine all the steady-state system size probabilities.

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